

# Effects of dipole-dipole interaction between cigar-like BECs of cold alkali atoms: Towards inverse square interactions

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(Dated: August 23, 2011)

We study the dipole-dipole interaction between ultracold alkali Bose atoms which, in general, was neglected because of the strong short range coupling in the alkali atoms. The heavy suppression of the short range coupling and enhancement of the dipole-dipole coupling between the cigar-like Bose-Einstein condensates (BECs) imply that the latter becomes dominant. This gives a proposal to observe the effect of this dipole-dipole interaction. In the limit of long length of the single BEC, the resulting effective one-dimensional models possess an effective inverse square interacting potential, the Calogero-Sutherland potential, which plays a fundamental role in many fields of the contemporary physics but their direct experimental realization challenges the experimentalists for a long time.

PACS numbers: 67.85.Pq, 37.10.Jk, 11.30.Pb

*Introduction* The dipole-dipole interactions between cold atoms/molecules play an important role in quantum simulations of strong correlated systems with cold atom/molecule gases [1, 2]. Most of these studies are for the polar molecules [3], the atoms with relevant strong dipolar couplings [4], and cold Rydberg atoms [2]. The effect of the dipole-dipole interaction between alkali atoms was thought of negligible because their dipolar coupling is too weak to compare with their  $s$ -wave scattering.

Experimentally, the cold alkali atom gases are much easier to be made and manipulated, and are more stable than those polar molecule and dipolar atom gases. An interesting question then is: Is there any way to change the strength of both kinds of interactions between alkali atoms so that the dipole-dipole interaction becomes dominant or the  $s$ -wave scattering becomes negligible? If so, we can use the alkali atom gases to study the dipole-dipole interactions, which have a great experimental advantage over using the other less stable cold atom/molecule systems.

One way to reduce the  $s$ -wave scattering is tuning the  $s$ -wave interaction to zero using a magnetic Feshbach resonance and then one can measure a very weak dipole-dipole interaction [5]. Another way to reduce the  $s$ -wave scattering is to consider the effective interaction between the single mode fluctuations in an array of cigar-like cold alkali atom Bose-Einstein condensates (BECs) which are confined in a one-dimensional optical lattice [6, 7]. How does the dipole-dipole interaction change for these single modes? This will be main topic in this paper. We will show an amazing result, i.e., the dipole-dipole interaction between atoms may reduce to an inverse square interaction between the single modes while the strength of the coupling constant is enhanced proportional to the atom number in a single BEC. On the other hand, the effective coupling of the  $s$ -wave scattering is reduced roughly as the inverse of the atom number in a single BEC. Therefore, in the limit of long length of the single BEC, the

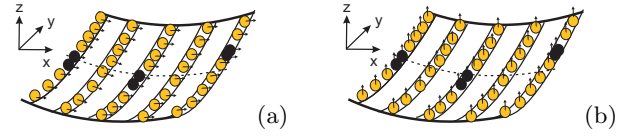


FIG. 1: (color online) The array of cigar-like cold atom clouds in an optical lattice. (a) The dipoles are polarized along the  $x$ -directions. (b) The dipoles are polarized along the  $z$ -directions. The yellow spots are the BEC substrate and the black are the quasi-atom with an inverse-square interaction.

effective on-site interaction can be neglected and the system becomes a pure inverse square interacting one.

One-dimensional inverse square interacting many-body systems, starting from Calogero-Sutherland (C-S) models [8, 9], nowadays play a fundamental role in many branches of modern physics [10, 11]. Besides their interest and importance in theoretical and mathematical physics, the C-S-type models are often applied to explain the physical phenomena in fractional quantum Hall effects [12], quantum exclusion statistics [13], and black hole physics [14], and so on. However, the existence of the inverse square interaction lacks a direct experimental realization. Our proposals in this paper present a possibility to realize them in laboratories.

We consider a cold Bose atom cloud with dipolar interactions. With a very strong  $z$ -direction confinement, the cloud has a pancake-like shape trapped in the  $x$ - $y$  plane with trapping frequencies  $\omega_x$  and  $\omega_y$ . Using a periodic optical potential along the  $x$ -direction, the pancake is incised into a lattice of cigar-like gases. An applied external field parallel to the  $x$ -axes, or perpendicular to this  $x$ - $y$  plane polarizes all dipoles parallel to the external field. (See Fig. 1(a) or (b), respectively). This setup is towards the Calogero model [8].

Another setup is rolling the  $x$ - $y$  plane to the surface of a cylinder. This can be obtained by confining a magnetic

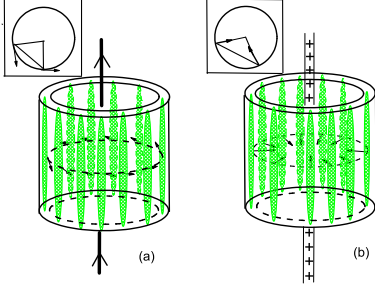


FIG. 2: (color online) The cigar-like BEC (green) are confined in a cylinder. (a) The magnetic dipoles; (b) The electric dipoles. The inserts indicate the directions of the dipoles.

dipolar BEC cloud to the surface of the cylinder with a stable current passing through the axis or an electric dipolar BEC cloud with a charged bar placed in the axis. (See Fig.2(a) or (b), respectively.) Sutherland's periodic variant [9] can be realized with this setup. A close relative of such a setup, cold atoms in a ring-shaped optical lattice, has been experimentally realized [15].

*Reduce to a single-band model* We first focus on the setup of Fig. 1 and set  $\omega_x = 0$ . The energy scale in the problem is  $\hbar\omega_y \ll k_B T \ll \hbar\omega_p$  where  $\omega_p$  is the frequency of lattice potential along the  $x$ -direction. Thus, we face a multi-band problem. Following the two-step procedure proposed in [7], we reduce the multi-band problem to an effective single-band one. First, we treat a single cigar-like gas in a given lattice site. The lowest energy level is a BEC  $\Psi_0(y)$  and higher levels are the thermal cloud. The equation of state with only short range interactions has been exactly solved in the weakly interacting limit [6]. It shows that even if there are phase fluctuations (collective modes), the atom number in a BEC may be determined at every given temperature [7]. For the atom carrying a dipole, besides the short range interaction, two dipoles located in  $\mathbf{r}$  and  $\mathbf{r}'$  interact as

$$V_d(\mathbf{r}, \mathbf{r}') = \frac{\mathbf{d} \cdot \mathbf{d}' - 3(\mathbf{d} \cdot \hat{\mathbf{R}})(\mathbf{d}' \cdot \hat{\mathbf{R}})}{R^3} \quad (1)$$

where  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$  and  $\hat{\mathbf{R}} = \mathbf{R}/R$ ;  $\mathbf{d}$  and  $\mathbf{d}'$  are the dipole moments of atoms located at  $\mathbf{r}$  and  $\mathbf{r}'$ , respectively. We consider all  $\mathbf{d}$  are parallel as shown in Fig. 1. The Fourier components of the dipolar interacting potential do not contain any singularity in the momentum space but only provide an anisotropic term. Therefore, for a cold atom gas with weak dipolar interaction, we can also use the method given in Ref. [6] to solve the equation of state and determine the atom number  $N_0(T)$  in a BEC.

The second step is to consider the coupling between sites [7]. If  $\hbar\omega_y \ll N_y U_0 \ll \hbar\omega_p$ , where  $N_y$  is the average number in a cigar-like gas and  $U_0$  the on-site repulsion, the wave function may be approximated by the

product of the single atom ground state wave function in the  $x$ -direction and the cigar-like atom gas in the  $y$  direction. In this approximation, the atomic field operator may be written as  $\psi(\mathbf{r}) = \sum_{i,\nu} a_{i,\nu} w_\nu(x - x_i) \Psi_\nu(y)$ , where  $\nu$  is the band index;  $w_\nu(x)$  is the Wannier function along the  $x$ -direction and  $\Psi_\nu(y)$  are the wave functions in the  $y$ -direction for the  $\nu$ -th band.  $a_{i,\nu}$  is the corresponding annihilation operator for a Wannier state in the  $\nu$ -th band at each site.

Because we have a BEC  $\Psi_0(y)$  at every site and  $N_0 \lesssim N_y$ , the coupling between sites will be dominated by tunneling from a BEC to condensate as opposed to a BEC to thermal cloud. The multi-band problem may be reduced to a single band one [7], i.e., the field operator may be approximated by  $\psi(\mathbf{r}) \approx \sum_i a_i w(x - x_i) \Psi_0(y)$ , where the bosonic mode  $a_i \equiv a_{i,0}$  corresponds to annihilating a quasi-atom mode described by the product of the Wannier function  $w(x) \equiv w_0(x)$  and the cigar-like BEC wave function. With this approximation, we can reduce the multi-band problem to an effective single band boson model with both renormalized on-site and renormalized dipole-dipole interactions. The renormalized hopping  $t_R$  and interaction  $U_R$  can be estimated in the Thomas-Fermi approximation [7]. The hopping  $t$  may not be renormalized [16] while  $U_R$  is reduced from its bare value considerably due to the repulsive on-site interactions that spread out the condensate wave function. The consistent condition for this reduction of  $U_R$  is  $l_p/a \ll N_0 \ll (\hbar\omega_y/\hbar\omega_p)^2 \sqrt{2\pi} L_y/a$ , where  $a$  is the  $s$ -wave scattering length.  $l_p$  is the lattice spacing and  $\hbar\omega_p = \hbar^2/m l_p^2$ ; and  $L_{x,y}$  are defined by  $\sqrt{\hbar/m\omega_{x,y}}$ .

*The lattice model with an inverse square interaction* The effective single-band lattice model can be obtained by keeping only the nearest neighbor hopping and expanding on-site energy to the second order near mean occupation  $N_0$  [7, 16], i.e.,

$$H_0 = - \sum_{\langle ij \rangle} t a_i^\dagger a_j + \frac{U_R}{2} \sum_i \delta n_i (\delta n_i - 1) + \sum_{i < j} U_{d,ij} \delta n_i \delta n_j, \quad (2)$$

where  $\delta n_i = a_i^\dagger a_i - N_0$  is the deviation of the atom number from the average number per site;  $t = \int dx w^*(x - x_i) (-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_p(x)) w(x - x_j)$  for a pair of nearest neighbor sites. We emphasize that here the hopping 'particle' is not the whole cigar-like BEC as a 'giant particle' but only the particle number variance between the adjacent lattice sites. The renormalized on-site potential  $U_R$  is given by  $U_0 \frac{l_p}{R_y} = \frac{4\pi\hbar^2 a}{m} \frac{l_p}{R_y}$  [7]. Hereafter, we take  $l_p = 1$  as the unit of length except explicitly stating. The renormalized dipolar interaction potential is given by

$$U_{d,ij} = \int d\mathbf{r} d\mathbf{r}' |w(x - x_i)|^2 |w(x' - x_j)|^2 V_d(\mathbf{r}, \mathbf{r}') |\Psi_0(y)|^2 |\Psi_0(y')|^2. \quad (3)$$

We now assume the condensate size  $R_y$  (the Thomas-Fermi radius of the cigar-like BEC in the  $y$ -direction) is much larger than the lattice size  $L_x$ . Then, one can use the approximation  $|\Psi_0(y)|^2 \approx N_0$ , the average density along in a single BEC and  $R_y \gg L_x$ . Under these approximations, it is easy to integrate over  $y$  and  $y'$  and arrives at

$$U_{d,ij} \approx \int dx dx' |w(x - x_i)|^2 |w(x' - x_j)|^2 \quad (4)$$

$$\times \frac{GR_y}{(x - x')^2 \sqrt{(x - x')^2 + R_y^2}} \approx \frac{G}{(x_i - x_j)^2} + O\left(\frac{1}{R_y^2}\right).$$

where  $G = -4d^2 N_0$  for Fig. 1(a) and  $G = 2d^2 N_0$  for Fig. 1(b). Thus, we have a one-dimensional lattice model with the on-site and inverse square interactions between the single modes from the ground state. We see that the finite size effect of the cigar-like BEC can only supply a negligible  $O(\frac{1}{R_y^2})$  correction.

*Domination of the renormalized dipole-dipole interaction* The stability condition of a single BEC at a given site is given by  $\epsilon_{dd} = d^2/(3U_0) < 1$  [17]. For the alkali metal atoms,  $d \sim 1\mu_B$  and  $\epsilon_{dd} \ll 1$ , e.g.,  $\epsilon_{dd} = 0.007$  for  $^{87}\text{Rb}$ . While the renormalized on-site interaction  $U_R$  is reduced by a factor  $\frac{l_p}{R_y}$  from  $U_0 = \frac{4\pi\hbar^2 a}{m}$  [7],  $d$  is enhanced to  $d_R = d\sqrt{N_0}$  as shown in (4). Thus, although  $\epsilon_{dd} \ll 1$ , the renormalized on-site interaction may be much weaker than the renormalized dipole-dipole interaction, i.e., the renormalized  $\epsilon_{d_R d_R} = d_R^2/(3U_R) = \epsilon_{dd} N_0 R_y / l_p$ , may be much larger than the unit. For example, if  $N_0 \sim 10^3$  and  $R_y / l_p > 50$  for  $^{87}\text{Rb}$ ,  $\epsilon_{d_R d_R} \gtrsim 10^3$ . Therefore, in the effective one-dimensional single-band model, the renormalized dipole-dipole interaction dominates and the renormalized on-site interaction may be neglected.

*Trapping potential in the  $x$ -direction* We have obtained a homogeneous lattice boson model with inverse square interaction by neglecting the trapping in the  $x$ -direction. Adding back the harmonic trap with the frequency  $\omega_x$  may affect the atom number  $N_0(x)$  at each site. If the width of a single BEC is wider, the number  $N_0$  of atom per site may vary site by site and  $N_0$  in the center of trapping may be several times of  $N_0$  in far away from the center. Within a cigar-like single dipolar BEC, the dipole-dipole repulsion is strong enough against raising the width of the single BEC at the trapping center. Therefore, the factorization of the ground state wave is still valid and  $N_0$  keeps an constant. The resulting inhomogeneous model is  $H$  in (2) added by a term  $\sum_i V_{h,i} \delta n_i$  where

$$V_{h,i} = \frac{1}{2} m \omega_x^2 \int dx |w(x - x_i)|^2 x^2 \quad (5)$$

is the harmonic trapping potential along the  $x$ -direction.

*Dilute gas limit and C-S model* Assuming the fluctuation atom number  $\delta N = \sum_i \delta n_i$  is fixed. In the dilute gas

limit with  $\delta N / L_x \ll 1$ , the  $\delta N$ -particle system may be described by a continuous model because the dispersion  $-t \cos k / \hbar \sim k^2 / (2m_R)$  with  $m_R \sim \hbar^2 / t$ . Neglecting the on-site interaction as argued above, the effective Hamiltonian reads

$$H_{CS} = \sum_{i=1}^{\delta N} \left( -\frac{\hbar^2}{2m_R} \frac{d^2}{dx_i^2} + \frac{m_R \omega_R^2}{2} x_i^2 \right) + \sum_{i < j} \frac{G}{|x_{ij}|^2} \quad (6)$$

where  $x_{ij} = x_j - x_i$  and the renormalized trapping potential is defined by  $m \omega_x^2 = m_R \omega_R^2$ . The model described by the Hamiltonian (6) is the famous C-S model [8, 9]. A bosonic 'particle' at  $x_i$  in this continuous model corresponds to one more atom than  $N_0$  at the lattice site  $i$  in the lattice model. The ground state wave function is given by

$$\Psi_{0,\lambda}(x_1, \dots, x_{\delta N}) = \prod_{1 \leq j < k \leq \delta N} |x_{jk}|^\lambda e^{-\frac{m_R \omega_R}{2\hbar} \sum_j x_j^2} \quad (7)$$

with the ground state energy  $E_g = \frac{1}{2} \delta N \hbar \omega_R (\lambda(\delta N - 1) + 1)$ . Here,  $\lambda$  is the solution of the equation  $\lambda(\lambda - 1) = g = (m_R / \hbar^2) G$ , e.g.,  $g \sim 0.021 N_0 U_0 / t$  for the repulsive interacting  $^{87}\text{Rb}$  atoms. This gives  $\lambda_{\pm} = \frac{1 \pm \sqrt{1+4g}}{2}$ . For a weak on-site interaction with  $U_0 / t < 1$  or  $\ll 1$ ,  $g$  may vary as  $t$  and is of the order  $O(1)$  and so is  $\lambda$  if  $N_0$  is fixed to the order of  $10^3$ .

For the setups in Fig. 1(a),  $g < 0$ . If  $g < -1/4$ ,  $\lambda$  becomes imaginary. This gives an imaginary ground state energy and the system is not stable against the collapse. However, when  $-1/4 \leq g < 0$ , the ground state is stable with  $0 \leq \lambda_- < 1/2$  although the interaction term is formally negative.

For the setups in Fig. 1(b),  $g > 0$ . For  $-\frac{1}{2} < \lambda_- < 0$ , i.e.,  $g < \frac{3}{4}$ , the wave function is still square integrable. This implies if the repulsive between particles is not strong enough, the system will also collapse. For a repulsion with  $g > \frac{3}{4}$ , (7) is no longer square integrable for  $\lambda = \lambda_- < -\frac{1}{2}$  and a physically stable solution is given by  $\lambda = \lambda_+$  which is larger than 1 here.

*Periodic interaction potential* We now consider the geometry in Fig. 2. In this geometry, the cylinder extends along the  $y$ -direction and circles in the  $x$ -direction. If  $R_y \gg L_x$ , one may prove that, up to a constant terms, the renormalized dipole-dipole interaction is a periodic inverse square interaction [9]

$$V \approx a_{\pm} \sum_{i \neq j} \frac{\pi^2 \lambda (\lambda - 1)}{L_x^2 \sin^2[\pi x_{ij} / L_x]} + O\left(\frac{1}{R_y^2}\right). \quad (8)$$

where  $a_- = -2$  and  $a_+ = 1$  with respect to Fig. 2(a) and for Fig. 2(b). The similar but periodic solutions were obtained by Sutherland [9].

*Experimental implications* The first effect from the dipole-dipole interaction between the alkali atoms is the collapse of the single modes to a single lattice site (a given

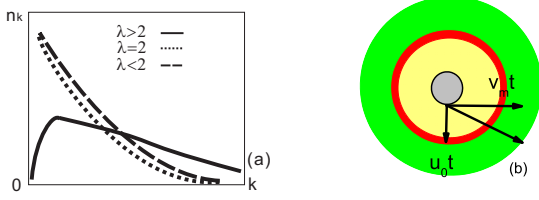


FIG. 3: (color online) (a) The schematic diagram of the momentum distribution. (b) The top-view of the time-of-flight image. The grey area is the original cylinder. The yellow only has very few atoms. Most of atoms arrive at the time  $t$  in the red area and the green is of those atoms with the momentum distribution (10).

single cigar-like BEC), when  $g < -1/4$  for the setups of Fig. 1(a) and Fig. 2(a) and  $g < 3/4$  for the setups of Fig. 1(b) and Fig. 2(b). This will never happen for a system dominated by the on-site repulsive interaction. The time-of-flight experiments measures the momentum distribution of the trapped cloud [18]. In these experiments, this collapse can be observed and can be thought as a definitive evidence of the effect of the renormalized dipole-dipole interaction.

In addition to this definitive evidence for the effect of the dipole-dipole interaction in alkali atoms, one can measure more observable of the systems. For example, one can consider the long wave length limit of the Green's function at  $t = 0$  of the Sutherland model, which is given by [19]

$$G(x, 0) \propto 1 + 2 \sum_{m=1}^{\infty} (-1)^m B_m \frac{\cos 2mk_F x}{|2k_F x|^{2m^2/\lambda}} \quad (9)$$

where  $B_m$  are regularization-dependent constants and  $k_F = \pi\delta N/L_x$  is the effective Fermi momentum. The momentum distribution is the Fourier transformation of this Green's function. For the lowest energy sector with  $m = 1$ , the momentum distribution for small  $k$  and large  $k$  is approximated by [19]

$$n_k \propto |k|^{\lambda/2-1} \text{ if } |k| \ll k_F \text{ or } \propto |k|^{-2\lambda-2} \text{ if } |k| \gg k_F \quad (10)$$

We see that when  $\lambda < 2$ , the system is in a quasi-condensed state while the momentum distribution vanishes in a power law for small  $k$  when  $\lambda > 2$ , a Luttinger liquid behavior. At the critical point  $\lambda = 2$ , a logarithmic divergence exists, i.e.,  $n_k = (1/2) \ln(2k_F/|k|)$  for  $|k| < 2k_F$  and  $n_k = 0$  for  $|k| \geq 2k_F$  [9]. Graphically, these momentum distributions have been shown in Ref. [19] and we give them schematically in Fig. 3(a).

The projected density profile in the plane perpendicular to the  $y$ -axis may directly reflect the momentum distribution  $n_k$  since the momentum along the circle is a good quantum number. The quasi-condensate is not

easy to be distinguish from the case with no dipolar interaction. However, for  $\lambda > 2$ , we switch off all traps but the inner wall of the cylinder. Because the cloud along the radial direction is the most squeezed, its expansion along this direction is much faster than that along the axial and tangent directions, owing to a larger zero point energy. Assume  $u_0$  is the velocity of the radial motion of a single cigar-like cloud and  $v_m$  is the fastest velocity of the particles along the circle. Let us see the top view of the cylinder. At the time  $t$  after turning off the traps, most of background BEC atoms arrive at the red area in Fig. 3(b) while the momentum distribution in Eq. (10) can be figured out from the atoms arrive at the green.

To match the theoretical results with the experimental data more precisely, the Green's function (9) and the momentum distribution (10) should be calculated in term of the lattice model (2). However, we believe Eq. (10) is qualitatively correct if  $G$  is in the designed region.

In conclusions, we showed the weak dipole-dipole interaction between the alkali atoms may renormalize to an inverse square potential dominating over the renormalized on-site interaction even without changing the bare on-site interaction by any external field. On the other hand, our proposal raised the possibility to implement the C-S model in nature, which was long expected but not directly realized yet in laboratories.

The author thanks Yingmei Liu, Kun Yang, Su Yi and Li You for useful discussions. This work was supported by National Natural Science Foundation of China, the national program for basic research of MOST of China.

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